

Remarks on the Clark theorem

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Abstract. The Clark theorem is important in critical point theory. For a class of even functionals it ensures the existence of infinitely many negative critical values converging to 0 and it has important applications to sublinear elliptic problems. We study the convergence of the corresponding critical points and we give a characterization of accumulation points of critical points together with examples, in which critical points with negative critical values converges to non-zero critical point. Our results improve the abstract results in Kajikiya [**Ka1**] and Liu-Wang [**LW**].

1. Introduction and main results

The Clark theorem is one of the most important results in critical point theory (Clark [**Cl**], see also Heinz [**H**]). It was successfully applied to sublinear elliptic problems with odd symmetry and the existence of infinitely many solutions which accumulate to 0 was shown.

To state the Clark theorem, we need some terminologies: let $(X, \|\cdot\|_X)$ be a Banach space and $I \in C^1(X, \mathbf{R})$.

- (i) For $c \in \mathbf{R}$ we say that $I(u)$ satisfies the $(PS)_c$ condition if any sequence $(u_j)_{j=1}^\infty \subset X$ with $I(u_j) \rightarrow c$, $\|I'(u_j)\|_{X^*} \rightarrow 0$ has a convergent subsequence.
- (ii) Let \mathcal{E} be the family of sets $A \subset X \setminus \{0\}$ such that A is closed and symmetric with respect to 0. For $A \in \mathcal{E}$, the genus $\gamma(A)$ is introduced by Krasnosel'skii [**Kr**] (c.f. Coffman [**Co**], Rabinowitz [**R**]) as the smallest integer n such that there exists an odd

continuous map $\zeta \in C(A, \mathbf{R}^n \setminus \{0\})$. When there does not exist such a map, we set $\gamma(A) = \infty$. See Rabinowitz [R] for fundamental properties of the genus.

Now we give a variant of the Clark theorem due to Heinz [H].

Theorem 1.1 (Heinz [H]). *Let $(X, \|\cdot\|_X)$ be a Banach space and suppose that $I(u) \in C^1(X, \mathbf{R})$ satisfies the following conditions:*

- (A1) $I(0) = 0$. $I(u)$ is even in u and bounded from below;
- (A2) $I(u)$ satisfies $(PS)_c$ for all $c < 0$;
- (A3) For any $k \in \mathbf{N}$, there exists $A \in \mathcal{E}$ such that

$$\gamma(A) \geq k \quad \text{and} \quad \sup_{u \in A} I(u) < 0.$$

Then $I(u)$ has a sequence $(c_j)_{j=1}^\infty$ of critical values of $I(u)$ such that

$$\begin{aligned} c_j &< 0 \quad \text{for all } j \in \mathbf{N}, \\ c_j &\rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Here

$$c_j = \inf_{A \in \mathcal{E}, \gamma(A) \geq j} \sup_{u \in A} I(u). \tag{1.1}$$

Remark 1.2. In [H], it was assumed that

(A2') $I(u)$ satisfies $(PS)_c$ for all $c \in \mathbf{R}$.

From its proof, we can easily see that $(PS)_c$ just for $c < 0$ is enough for the existence of critical values.

By Theorem 1.1, there exists a sequence $(u_j)_{j=1}^\infty$ of critical points of $I(u)$ such that $I(u_j) = c_j \rightarrow -0$ as $j \rightarrow \infty$. Thus it is natural to ask whether $u_j \rightarrow 0$ holds or not. More generally, the existence of a sequence of non-zero critical points $(u_j)_{j=1}^\infty$ (or critical points with negative critical values) satisfying $u_j \rightarrow 0$ is of interest. This question has been studied by Kajikiya [Ka1] and Liu-Wang [LW] together with applications to sublinear elliptic problems. We note that Liu-Wang [LW] also studied periodic solutions of Hamiltonian systems. More precisely, under the assumptions of (A1), (A2') and (A3), Kajikiya [Ka1] showed either

- (C1) There exists a sequence $(u_j)_{j=1}^\infty$ such that

$$I'(u_j) = 0, \quad I(u_j) < 0 \quad \text{and} \quad u_j \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

or

(C2) There exists two sequences $(u_j)_{j=1}^\infty$ and $(v_j)_{j=1}^\infty$ such that

$$I'(u_j) = 0, \quad I(u_j) = 0, \quad u_j \neq 0 \text{ and } u_j \rightarrow 0 \text{ as } j \rightarrow \infty$$

and

$$I'(v_j) = 0, \quad I(v_j) < 0 \text{ and } v_j \text{ converges to a non-zero limit.}$$

holds.

Liu-Wang [LW] assumed (A1), (A2') and the following (A3'), which is stronger than (A3),

(A3') For any $k \in \mathbf{N}$ there exists a k -dimensional subspace X^k of X and $\rho_k > 0$ such that

$$\sup\{I(u); u \in X^k, \|u\|_X = \rho_k\} < 0$$

and they showed either (C1) above or

(C3) There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that

$$\|u\|_X = a \quad \text{and} \quad I(u) = 0.$$

In what follows, we denote by \widehat{K}_0 the connected component of $K_0 = \{u \in X; I'(u) = 0, I(u) = 0\}$ including 0.

Remark 1.3. From their proof of their main result, Liu-Wang [LW] claimed that (C3) can be strengthened as

(C3') There exists $r > 0$ such that

$$\widehat{K}_0 \cap \{u \in X; \|u\|_X = r\} \neq \emptyset.$$

The aim of this paper is to show the following Theorem 1.4 and Theorem 1.6; In Theorem 1.4, we give a new characterization of accumulation points of critical points with negative critical values and unifies the results of Kajikiya and Liu-Wang. On the other hand, in Theorem 1.6 we answer a natural question concerning (C1), which is stated below. We believe that Theorems 1.4 and 1.6 give us a better understanding of the Clark theorem.

First we give our Theorem 1.4.

Theorem 1.4. *Let $(X, \|\cdot\|_X)$ be a Banach space and suppose $I \in C^1(X, \mathbf{R})$ satisfies (A1), (A3) and (A2'') $I(u)$ satisfies $(PS)_c$ for all $c \leq 0$.*

Then there exists a sequence $(u_j)_{j=1}^\infty \subset X$ of critical points of $I(u)$ such that

$$I(u_j) < 0 \quad \text{for all } j \in \mathbf{N}, \quad (1.2)$$

$$I(u_j) \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (1.3)$$

$$\text{dist}(u_j, \widehat{K}_0) \left(\equiv \inf\{\|u_j - v\|_X; v \in \widehat{K}_0\} \right) \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

As an immediate corollary to our Theorem 1.4, we have

Corollary 1.5. *Under the assumptions of Theorem 1.4, assume that (C1) does not take place. Then $\widehat{K}_0 \neq \{0\}$.*

Since $\widehat{K}_0 \neq \{0\}$ implies (C2) and (C3), Corollary 1.5 covers the results of Kajikiya [Ka1] and Liu-Wang [LW].

Next we study a question concerning (C1). In many applications of the Clark theorem to sublinear elliptic equations, there exist sequences $(u_j)_{j=1}^\infty$ of solutions with (1.2), (1.3) and

$$u_j \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (1.4)$$

So (C1) may be expected under the assumption of Theorem 1.4 and a natural question is to ask whether (C1) always takes place under the assumption of Theorem 1.4 or not. Our Theorem 1.6 answers this question negatively.

Theorem 1.6. *Conditions (A1), (A2''), (A3') do not imply (C1). In particular, under the assumptions of Theorem 1.4, (C1) does not hold in general.*

Remark 1.7. An example related to our Theorem 1.6 was given in Example 1.3 of [Ka1] (c.f. [Ka2]). It shows that there exists a functional $I \in C^1(X, \mathbf{R})$ which satisfies (A1), (A2''), (A3) and the following property:

There exists an $r_0 > 0$ independent of j such that

$$I'(u) = 0 \text{ and } I(u) = c_j \quad \text{imply} \quad \|u\|_X \geq r_0.$$

Here c_j is given in (1.1) and c_j satisfies $c_j < 0$ and $c_j \rightarrow 0$ as $j \rightarrow \infty$. Thus a special case of (C1) does not hold for I . In Section 3.1 we give another example $I \in C^1(\ell^2, \mathbf{R})$

for which we give an explicit description of all critical points of $I(u)$ and no critical points with negative critical values do not exist in a neighborhood of 0. Especially (C1) does not hold for our $I(u)$. Our example also shows a typical situation of our Theorem 1.4.

Finally we remark that in our Theorem 1.4, (A2''), especially $(PS)_0$ is important. In fact, we have

Theorem 1.8. *Under the assumptions of Theorem 1.1, especially without $(PS)_0$, the conclusion of Theorem 1.4 does not hold in general.*

In the following Section 2, we give a proof to our Theorem 1.4. Here estimates of $I'(u)$ play important roles. In Section 3, we give two examples which show Theorems 1.6 and 1.8.

2. Proof of Theorem 1.4

In what follows, we use the following notation for $\delta > 0$

$$\begin{aligned} B_\delta(u) &= \{x \in X; \|x - u\|_X < \delta\} \quad \text{for } u \in X, \\ N_\delta(D) &= \{x \in X; \text{dist}(x, D) < \delta\} \quad \text{for } D \subset X, \end{aligned}$$

where

$$\text{dist}(x, D) = \inf_{y \in D} \|x - y\|_X.$$

We note that $N_\delta(D) = \bigcup_{y \in D} B_\delta(y)$.

2.1. A fundamental fact from topology

To show our Theorem 1.4, we need the following characterization of connected components of compact sets.

Lemma 2.1. *Let $D \subset X$ be a compact set such that $0 \in D$. For $\delta > 0$, let O_δ be the connected component of $N_\delta(D)$ including 0. Then we have*

$$\bigcap_{\delta > 0} \overline{O_\delta} = \widehat{D},$$

where \widehat{D} is the connected component of D including 0.

Proof. By the definition of O_δ and \widehat{D} , it is clear that $\widehat{D} \subset O_\delta$ for all $\delta > 0$. Thus

$$\widehat{D} \subset \bigcap_{\delta > 0} O_\delta \subset \bigcap_{\delta > 0} \overline{O_\delta}.$$

By the compactness of D , we also have $D = \bigcap_{\delta>0} \overline{N_\delta(D)}$.

We set

$$A = \bigcap_{\delta>0} \overline{O_\delta} \subset D.$$

It suffices to show that A is connected. For $\delta > 0$ we also set $D_\delta = \overline{O_\delta} \cap D$. Then we have

$$A = \bigcap_{\delta>0} D_\delta, \tag{2.1}$$

$$O_\delta = \bigcup_{u \in D_\delta} B_\delta(u), \tag{2.2}$$

$$\delta_1 < \delta_2 \text{ implies } D_{\delta_1} \subset D_{\delta_2}. \tag{2.3}$$

Arguing indirectly, we suppose that A is not connected. Then there exist two compact sets $A_1, A_2 \subset X$ such that $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2 = A$. We set

$$\beta = \frac{1}{2} \text{dist}(A_1, A_2) > 0.$$

For each $\delta > 0$, since O_δ is a connected set including $A_1 \cup A_2$, we have

$$O_\delta \cap \{x \in X; \text{dist}(x, A_1) = \beta\} \neq \emptyset.$$

By (2.2), we can see that for any $x \in O_\delta$ there exists $u \in D_\delta$ such that $x \in B_\delta(u)$. Thus

$$D_\delta \cap \{x \in X; \text{dist}(x, A_1) \in [\beta - \delta, \beta + \delta]\} \neq \emptyset.$$

Since $\left(D_\delta \cap \{x \in X; \text{dist}(x, A_1) \in [\beta - \delta, \beta + \delta]\}\right)_{\delta>0}$ has the finite intersection property by (2.3), we have

$$\begin{aligned} & A \cap \{x \in X; \text{dist}(x, A_1) = \beta\} \\ &= \bigcap_{\delta>0} \left(D_\delta \cap \{x \in X; \text{dist}(x, A_1) \in [\beta - \delta, \beta + \delta]\}\right) \neq \emptyset, \end{aligned}$$

which contradicts with the choice of $\beta > 0$. Thus A is a connected set. ■

2.2. A gradient estimate

Suppose that $I(u) \in C^1(X, \mathbf{R})$ satisfies the assumptions of Theorem 1.4. We use the following notation:

$$K = \{u \in X; I'(u) = 0\},$$

$$K_0 = \{u \in K; I(u) = 0\},$$

$$K_- = \{u \in X; \text{there exists } (v_j)_{j=1}^\infty \subset K \text{ such that}$$

$$I(v_j) < 0 \text{ for all } j \text{ and } I(v_j) \rightarrow 0, v_j \rightarrow u \text{ as } j \rightarrow \infty\}.$$

By $(PS)_0$, we have $K_- \subset K_0$. We also use notation for $a < b$

$$\begin{aligned} [I \leq a] &= \{u \in X; I(u) \leq a\}, \\ [a \leq I \leq b] &= \{u \in X; a \leq I(u) \leq b\}. \end{aligned}$$

It is clear that $0 \in K_0$. We denote by \widehat{K}_0 the connected component of K_0 including 0. To show our Theorem 1.4 it suffices to prove

$$K_- \cap \widehat{K}_0 \neq \emptyset. \quad (2.4)$$

For $\delta > 0$, let O_δ be the connected component of $N_\delta(K_0)$ including 0. By Lemma 2.2, we have

$$\widehat{K}_0 = \bigcap_{\delta > 0} \overline{O_\delta}.$$

Thus to prove (2.4) it suffices to show

$$\overline{O_\delta} \cap K_- \neq \emptyset \quad \text{for all } \delta > 0.$$

We argue indirectly and suppose for some $\delta_0 > 0$

$$\overline{O_{\delta_0}} \cap K_- = \emptyset. \quad (2.5)$$

Under the assumption (2.5), we set

$$K_{0,i} = \overline{O_{\delta_0}} \cap K_0, \quad K_{0,e} = K_0 \setminus O_{\delta_0}.$$

Then $K_{0,i}$ and $K_{0,e}$ are disjoint compact sets such that $K_0 = K_{0,i} \cup K_{0,e}$ and

$$\text{dist}(K_{0,i}, K_{0,e}) \geq 2\delta_0, \quad (2.6)$$

$$K_- \subset K_{0,e}. \quad (2.7)$$

We note that (2.7) follows from (2.5).

First we have

Lemma 2.2. *Assume (2.5). Then for any $r > 0$ there exist $\rho > 0$ and $\nu > 0$ such that*

$$[-\rho \leq I \leq 0] \cap K \subset N_r(K_0), \quad (2.8)$$

$$[-\rho \leq I < 0] \cap K \subset N_r(K_-) \subset N_r(K_{0,e}), \quad (2.9)$$

$$\|I'(u)\|_{X^*} \geq \nu \quad \text{for all } u \in [-\rho \leq I \leq 0] \setminus N_r(K_0). \quad (2.10)$$

Moreover for any $\varepsilon \in (0, \rho)$ there exists $\nu_\varepsilon \in (0, \nu]$ such that

$$\|I'(u)\|_{X^*} \geq \nu_\varepsilon \quad \text{for all } u \in [-\rho \leq I \leq -\varepsilon] \setminus N_r(K_{0,e}). \quad (2.11)$$

Proof. Using $(PS)_0$ and the definition of K_- , we can check (2.8)–(2.10) easily for small ρ and $\nu > 0$. We show (2.11). Suppose that for $r, \rho, \nu > 0$, (2.8)–(2.10) hold. If (2.11) does not hold, we can find $\varepsilon \in (0, \rho)$ and a sequence $(u_j)_{j=1}^\infty$ such that

$$I(u_j) \in [-\rho, -\varepsilon], \quad \|I'(u_j)\| \rightarrow 0, \quad u_j \notin N_r(K_{0,e}),$$

By (PS) , we can extract a subsequence (u_{j_k}) such that $u_{j_k} \rightarrow u_0$ for some $u_0 \in [-\rho \leq I \leq -\varepsilon] \cap K$. By (2.9), we have $u_{j_k} \notin N_r(K_{0,e})$ for large k , which is a contradiction. Thus we have (2.11). ■

2.3. Deformation argument

The aim of this section is the following

Proposition 2.3. *Assume (2.5). Then for any $r \in (0, \delta_0/3]$ there exists $d > 0$ with the following property: for any $\varepsilon \in (0, d/2]$ there exists an odd continuous map $\eta_\varepsilon : [I < 0] \rightarrow [I < 0]$ such that*

$$\eta_\varepsilon([I \leq -\varepsilon]) \subset [I \leq -d] \cup N_{3r}(K_{0,e}).$$

Proof. First we define an ODE in X to define η_ε .

For a given $r \in (0, \delta_0/3]$, let $\rho, \nu > 0$ be constants given in Lemma 2.2. We set

$$d = \frac{1}{3} \min\{\rho, \nu r\} > 0. \quad (2.12)$$

Then again by Lemma 2.2, for any given $\varepsilon \in (0, d]$ there exists $\nu_\varepsilon > 0$ with the property (2.11).

By (2.9), we have $I'(u) \neq 0$ for all $u \in [-3d \leq I < 0] \setminus N_r(K_{0,e})$. Thus there exists a locally Lipschitz odd vector field $V(u) : [-3d \leq I < 0] \setminus N_r(K_{0,e}) \rightarrow X$ such that

$$\begin{aligned} \|V(u)\|_X &\leq 1 \quad \text{for all } u \in [-3d \leq I < 0] \setminus N_r(K_{0,e}), \\ I'(u)V(u) &> 0 \quad \text{for all } u \in [-3d \leq I < 0] \setminus N_r(K_{0,e}), \end{aligned} \quad (2.13)$$

$$I'(u)V(u) \geq \frac{\nu}{2} \quad \text{for all } u \in [-3d \leq I < 0] \setminus N_r(K_0), \quad (2.14)$$

$$I'(u)V(u) \geq \frac{\nu_\varepsilon}{2} \quad \text{for all } u \in [-3d \leq I \leq -\varepsilon] \setminus N_r(K_{0,e}). \quad (2.15)$$

Let $\phi_1(u), \phi_2(u) : X \rightarrow [0, 1]$ be even Lipschitz continuous functions such that

$$\phi_1(u) = \begin{cases} 1 & \text{for } u \in [-d \leq I], \\ 0 & \text{for } u \in [I \leq -2d], \end{cases} \quad \phi_2(u) = \begin{cases} 1 & \text{for } u \in X \setminus N_{2r}(K_{0,e}), \\ 0 & \text{for } u \in N_r(K_{0,e}). \end{cases}$$

We set

$$\tilde{V}(u) = \phi_1(u)\phi_2(u)V(u)$$

and we note that $\tilde{V}(u)$ is well-defined on $[I < 0]$. For $u \in [I < 0]$ we consider

$$\begin{cases} \frac{d\eta}{dt} = -\tilde{V}(\eta), \\ \eta(0, u) = u. \end{cases}$$

We have for all (t, u)

$$\eta(t, -u) = -\eta(t, u), \quad (2.16)$$

$$\left\| \frac{d\eta}{dt} \right\|_X \leq 1. \quad (2.17)$$

Since

$$\frac{d}{dt}I(\eta(t, u)) = I'(\eta(t, u))\frac{d\eta}{dt} = -I'(\eta)\tilde{V}(\eta),$$

it follows from (2.13)–(2.15) that

$$\frac{d}{dt}I(\eta(t, u)) \leq 0 \quad \text{if } \eta(t, u) \in [I < 0], \quad (2.18)$$

$$\frac{d}{dt}I(\eta(t, u)) \leq -\frac{\nu}{2} \quad \text{if } \eta(t, u) \in [-d \leq I < 0] \setminus N_{2r}(K_0), \quad (2.19)$$

$$\frac{d}{dt}I(\eta(t, u)) \leq -\frac{\nu_\varepsilon}{2} \quad \text{if } \eta(t, u) \in [-d \leq I \leq -\varepsilon] \setminus N_{2r}(K_{0,e}). \quad (2.20)$$

By (2.17) and (2.18), we note that for any $u \in [I < 0]$, $\eta(t, u)$ exists globally, that is, $\eta(t, u) : [0, \infty) \times [I < 0] \rightarrow [I < 0]$ is well-defined. For a latter use, we note that

$$\overline{N_{3r}(K_{0,e})} \setminus N_{2r}(K_{0,e}) \subset X \setminus N_{2r}(K_0).$$

Thus by (2.19)

$$\frac{d}{dt}I(\eta(t, u)) \leq -\frac{\nu}{2} \quad \text{if } \eta(t, u) \in [-d \leq I < 0] \cap (N_{3r}(K_{0,e}) \setminus N_{2r}(K_{0,e})). \quad (2.21)$$

Next we claim that

Claim. Let $T_\varepsilon = \frac{2d}{\nu_\varepsilon}$. Then

$$\eta(T_\varepsilon, u) \in [I \leq -d] \cup N_{3r}(K_{0,e}) \quad \text{for any } u \in [I \leq -\varepsilon]. \quad (2.22)$$

To prove (2.22), it suffices to show that if $u \in [I \leq -\varepsilon]$ satisfies

$$\eta(T_\varepsilon, u) \notin [I \leq -d], \quad (2.23)$$

then

$$\eta(T_\varepsilon, u) \in N_{3r}(K_{0,e}). \quad (2.24)$$

We note that under the condition (2.23)

$$\eta(t, u) \in [-d \leq I \leq -\varepsilon] \quad \text{for all } t \in [0, T_\varepsilon]. \quad (2.25)$$

Step 1: Assume (2.23), i.e., (2.25). Then

$$\eta([0, T_\varepsilon], u) \cap N_{2r}(K_{0,e}) \neq \emptyset. \quad (2.26)$$

In fact, if (2.26) does not hold, it follows from (2.20) that

$$\frac{d}{ds} I(\eta(s, u)) \leq -\frac{\nu_\varepsilon}{2} \quad \text{for all } s \in [0, T_\varepsilon].$$

Thus, by the definition of T_ε ,

$$\begin{aligned} I(\eta(T_\varepsilon, u)) &\leq I(u) + \int_0^{T_\varepsilon} \frac{d}{ds} I(\eta(s, u)) ds \\ &\leq -\varepsilon - \frac{\nu_\varepsilon}{2} T_\varepsilon \\ &< -d, \end{aligned}$$

which is in contradiction with (2.23).

Step 2: Assume (2.23), i.e., (2.25). Then (2.24) holds.

Assume (2.24) does not hold. Then $\eta(T_\varepsilon, u) \notin N_{3r}(K_{0,e})$ and by (2.26) the orbit $\eta(t, u)$ enters in $N_{2r}(K_{0,e})$ for some $t \in [0, T_\varepsilon]$. Thus there exists an interval $[t_0, t_1] \subset [0, T_\varepsilon]$ such that

$$\begin{aligned} \eta(t_0, u) &\in \partial N_{2r}(K_{0,e}), \\ \eta(t_1, u) &\in \partial N_{3r}(K_{0,e}), \\ \eta(t, u) &\in \overline{N_{3r}(K_{0,e})} \setminus N_{2r}(K_{0,e}) \quad \text{for all } t \in [t_0, t_1]. \end{aligned}$$

By (2.17), we have

$$\begin{aligned} r &\leq \|\eta(t_1, u) - \eta(t_0, u)\|_X \leq \int_{t_0}^{t_1} \left\| \frac{d}{ds} \eta(s, u) \right\|_X ds \\ &\leq t_1 - t_0. \end{aligned}$$

Thus by (2.21),

$$\begin{aligned}
I(\eta(T_\varepsilon, u)) &\leq I(u) + \int_0^{T_\varepsilon} \frac{d}{ds} I(\eta(s, u)) ds \\
&\leq I(u) + \int_{t_0}^{t_1} \frac{d}{ds} I(\eta(s, u)) ds \\
&\leq I(u) - \frac{\nu}{2}(t_1 - t_0) \\
&\leq -\varepsilon - \frac{\nu r}{2} \leq -d.
\end{aligned}$$

This is a contradiction to (2.23). Thus we have $\eta(T_\varepsilon, u) \in N_{3r}(K_{0,e})$ and the conclusion of Step 2 holds.

Setting $\eta_\varepsilon(u) = \eta(T_\varepsilon, u)$, we have the desired deformation. ■

2.4. End of the proof of Theorem 1.4

Proof of Theorem 1.4. Since $K_{0,e} \in \mathcal{E}$ is compact, we can see

$$\gamma_{0,e} \equiv \gamma(K_{0,e}) < \infty.$$

Moreover for small $r \in (0, \delta_0/3]$

$$\gamma(\overline{N_{3r}(K_{0,e})}) = \gamma(K_{0,e}) = \gamma_{0,e}.$$

We fix such an r and we choose $d > 0$ by Proposition 2.3.

By Clark's theorem [Cl], we have

$$c_k \equiv \inf_{\gamma(A) \geq k} \sup_{u \in A} I(u) \nearrow 0.$$

Thus there exists k_0 such that $c_{k_0} > -d$. That is, $\gamma([I \leq -d]) < k_0$. Thus

$$\gamma([I \leq -d] \cup \overline{N_{3r}(K_{0,e})}) \leq \gamma([I \leq -d]) + \gamma(\overline{N_{3r}(K_{0,e})}) \leq \gamma_{0,e} + k_0. \quad (2.27)$$

By the assumption of Theorem 1.4, there exists $A \in \mathcal{E}$ such that

$$\gamma(A) > \gamma_{0,e} + k_0 \quad \text{and} \quad \sup_{u \in A} I(u) < 0.$$

Choosing $\varepsilon \in (0, d)$ such that $\sup_{u \in A} I(u) < -\varepsilon$, we have

$$\gamma([I \leq -\varepsilon]) \geq \gamma(A) > \gamma_{0,e} + k_0. \quad (2.28)$$

On the other hand, by Proposition 2.3, there exists a continuous odd map $\eta_\varepsilon : [I < 0] \rightarrow [I < 0]$ such that

$$\eta_\varepsilon([I \leq -\varepsilon]) \subset [I \leq -d] \cup N_{3r}(K_{0,e}).$$

Thus by (2.27)

$$\begin{aligned}\gamma([I \leq -\varepsilon]) &\leq \gamma(\eta_\varepsilon([I \leq -\varepsilon])) \\ &\leq \gamma([I \leq -d] \cup \overline{N_{3r}(K_{0,e})}) \\ &\leq \gamma_{0,e} + k_0,\end{aligned}$$

which is in contradiction with (2.28). Thus (2.5) cannot take place and we complete the proof of our Theorem 1.4. ■

3. Some examples

In this section we give two examples which show Theorems 1.6 and 1.8.

3.1. An example which shows Theorem 1.6

We give an example which shows that (A1), (A2''), (A3') do not imply (C1). We work in the space $X = \ell^2$, that is,

$$\begin{aligned}X &= \left\{ (t, x_1, x_2, \dots); t, x_j \in \mathbf{R} \ (j = 1, 2, \dots), \ t^2 + \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}, \\ \|(t, x_1, x_2, \dots)\|_X &= \left(t^2 + \sum_{j=1}^{\infty} |x_j|^2 \right)^{1/2}.\end{aligned}$$

Since the first component has a special role in our argument, we use notation (t, x_1, x_2, \dots) for elements of X .

We consider a functional $I(t, x_1, x_2, \dots) : X \rightarrow \mathbf{R}$ in a form

$$I(t, x_1, x_2, \dots) = \frac{1}{2} \sum_{j=1}^{\infty} |x_j|^2 - \frac{2}{3} \sum_{j=1}^{\infty} 3^{-j} \left(a_+(t)(x_j)_+^{3/2} + a_-(t)(x_j)_-^{3/2} \right) + \varphi(t),$$

where $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$ and $a_+(t)$, $a_-(t) \in C^1(\mathbf{R}, \mathbf{R})$ are given by

$$a_+(t) = 2 + \mu(t), \quad a_-(t) = 2 - \mu(t).$$

Here $\mu(t) \in C^1(\mathbf{R}, \mathbf{R})$ satisfies

$$\mu(t) \in [-1, 1] \quad \text{for all } t \in \mathbf{R}, \tag{3.1}$$

$$\mu(-t) = -\mu(t) \quad \text{for all } t \in \mathbf{R}, \tag{3.2}$$

$$\mu(t) = 1 \quad \text{for } t \geq 1, \tag{3.3}$$

$$\mu(t) = -1 \quad \text{for } t \leq -1, \tag{3.4}$$

$$\mu'(t) > 0 \quad \text{for } t \in (-1, 1). \tag{3.5}$$

It follows from (3.1)–(3.5) that

$$a_+(t), a_-(t) \in [1, 3] \quad \text{for all } t \in \mathbf{R}, \quad (3.6)$$

$$a_+(-t) = a_-(t), \quad a_-(-t) = a_+(t) \quad \text{for all } t \in \mathbf{R}, \quad (3.7)$$

$$a'_+(t) = \mu'(t) > 0, \quad a'_-(t) = -\mu'(t) < 0 \quad \text{for all } t \in (-1, 1). \quad (3.8)$$

Finally we define $\varphi(t) \in C^1(\mathbf{R}, \mathbf{R})$ by

$$\varphi(t) = \begin{cases} (t-1)^2 & \text{for } t > 1, \\ 0 & \text{for } t \in [-1, 1], \\ (t+1)^2 & \text{for } t < -1. \end{cases}$$

We can see that $I(t, x_1, x_2, \dots)$ has the following properties.

Proposition 3.1.

- (i) $I(t, x_1, x_2, \dots) \in C^1(X, \mathbf{R})$;
- (ii) $I(t, x_1, x_2, \dots)$ is bounded from below and coercive on X ;
- (iii) $I(t, x_1, x_2, \dots)$ satisfies $(PS)_c$ for all $c \in \mathbf{R}$;
- (iv) $I(t, x_1, x_2, \dots)$ is even in (t, x_1, x_2, \dots) , that is, $I(-t, -x_1, -x_2, \dots) = I(t, x_1, x_2, \dots)$;
- (v) $I(t, x_1, x_2, \dots)$ satisfies (A3').

Proof. It follows from Hölder inequality that

$$\left| \sum_{j=1}^{\infty} b_j(x_j)^{3/2} \right| \leq \left(\sum_{j=1}^{\infty} b_j^4 \right)^{1/4} \left(\sum_{j=1}^{\infty} |x_j|^2 \right)^{3/4}, \quad (3.9)$$

from which we can see that $I(t, x_1, x_2, \dots)$ is well-defined as a functional on X . Using (3.9), we can also see (i)–(iii). (iv) follows from (3.7).

For $k \in \mathbf{N}$, setting $X^k = \{(0, x_1, x_2, \dots, x_k, 0, 0, \dots); (x_1, x_2, \dots, x_k) \in \mathbf{R}^k\}$, we can easily find $\rho_k > 0$ such that

$$\sup\{I(u); u \in X^k, \|u\|_X = \rho_k\} < 0.$$

Thus (v) holds. ■

By Proposition 3.1, we can apply the Clark Theorem to $I(t, x_1, x_2, \dots)$. On the other hand, we have

Proposition 3.2. *Let K be the set of all critical points of $I(t, x_1, x_2, \dots)$. Then we have*

$$K = Z \cup N \cup (-N), \quad (3.10)$$

where

$$\begin{aligned} Z &= \{(t, 0, 0, \dots); t \in [-1, 1]\}, \\ N &= \{(1, x_1, x_2, \dots); x_j \in \{0, 9 \cdot 3^{-2j}, -3^{-2j}\} \text{ for all } j\}. \end{aligned} \quad (3.11)$$

Moreover we have

$$I(u) = 0 \quad \text{for all } u \in Z, \quad (3.12)$$

$$I(u) < 0 \quad \text{for all } u \in K \setminus Z. \quad (3.13)$$

Proof. Since

$$\partial_t I(t, x_1, x_2, \dots) = \varphi'(t) \quad \text{for all } (t, x_1, x_2, \dots) \in X \text{ with } |t| \geq 1, \quad (3.14)$$

we see that $I'(t, x_1, x_2, \dots) = 0$ implies $t \in [-1, 1]$.

In what follows, we assume that (t, x_1, x_2, \dots) ($t \in [-1, 1]$) is a critical point of I and give more precise description. First we show

Step 1: For any $j \in \mathbf{N}$,

$$x_j = 0, \quad 3^{-2j} a_+(t)^2, \quad \text{or} \quad -3^{-2j} a_-(t)^2. \quad (3.15)$$

In particular, we have

(i) if $x_j \neq 0$,

$$3^{-2j} \leq |x_j| \leq 9 \cdot 3^{-2j}. \quad (3.16)$$

(ii) $K \cap \{t = 1\} = N$, $K \cap \{t = -1\} = -N$, where N is given in (3.11).

In fact, it follows from $\partial_{x_j} I(t, x_1, x_2, \dots) = 0$ that

$$x_j - 3^{-j} \left(a_+(t)(x_j)_+^{1/2} - a_-(t)(x_j)_-^{1/2} \right) = 0.$$

From which we can get (3.15). (i) and (ii) follow from the property (3.6) and $a_+(1) = a_-(-1) = 3$, $a_+(-1) = a_-(1) = 1$.

Step 2: When $t \in (-1, 1)$, it holds that $x_j = 0$ for all j .

It follows from $\partial_t I(t, x_1, x_2, \dots) = 0$ that

$$\sum_{j=1}^{\infty} 3^{-j} \left(a'_+(t)(x_j)_+^{3/2} + a'_-(t)(x_j)_-^{3/2} \right) = 0.$$

By (3.8),

$$\sum_{j=1}^{\infty} 3^{-j} \left((x_j)_+^{3/2} - (x_j)_-^{3/2} \right) = 0.$$

Arguing indirectly, we assume that $x_j \neq 0$ for some j and let j_0 be the smallest integer such that $x_j \neq 0$. Then we have

$$\begin{aligned} 3^{-j_0} |x_{j_0}|^{3/2} &= \left| 3^{-j_0} \left((x_{j_0})_+^{3/2} - (x_{j_0})_-^{3/2} \right) \right| = \left| \sum_{j=j_0+1}^{\infty} 3^{-j} \left((x_j)_+^{3/2} - (x_j)_-^{3/2} \right) \right| \\ &\leq \sum_{j=j_0+1}^{\infty} 3^{-j} |x_j|^{3/2}. \end{aligned} \quad (3.17)$$

By (3.16),

$$\text{the right hand side of (3.17)} \leq \sum_{j=j_0+1}^{\infty} 3^{-j} (9 \cdot 3^{-2j})^{3/2} = \sum_{j=j_0+1}^{\infty} 27 \cdot 3^{-4j} < \frac{2}{3} \cdot 3^{-4j_0}.$$

Again by (3.16),

$$\text{the left hand side of (3.17)} \geq 3^{-j_0} (3^{-2j_0})^{3/2} = 3^{-4j_0},$$

which is in contradiction with (3.17). Thus we have $x_j = 0$ for all $j \in \mathbb{N}$.

Step 3: Conclusion.

(3.10) follows from Steps 1–2. We can also verify (3.12)–(3.13) easily. ■

As an immediate corollary to Proposition 3.2, we have

Corollary 3.3.

- (i) *Points $(t, 0, \dots) \in X$ ($t \in (-1, 1)$) cannot be accumulation points of critical points with negative critical values.*
- (ii) *$(1, 0, 0, \dots), (-1, 0, 0, \dots) \in X$ are accumulation points of critical points with negative critical values.*

Thus Corollary 3.3 shows that (C1) does not hold in general under the conditions (A1), (A2''), (A3').

3.2. An example which shows Theorem 1.8

Next we give another example, which shows that without $(PS)_0$ the conclusion of Theorem 1.4 does not hold in general. Here we work in the Hilbert space $(E, \|\cdot\|_E)$ given by

$$\begin{aligned} E &= H_0^1(0, 1), \\ \|u\|_E &= \left(\int_0^1 |u_x|^2 dx \right)^{1/2} \quad \text{for } u \in E. \end{aligned}$$

For $p \in (0, 1)$ we define $J(u) \in C^1(E, \mathbf{R})$ by

$$J(u) = \frac{1}{2} \|u\|_E^2 - \frac{1}{p+1} \int_0^1 |u|^{p+1} dx : E \rightarrow \mathbf{R}.$$

Critical points of $J(u)$ are solutions of the following sublinear elliptic equation:

$$\begin{cases} u_{xx} + |u|^{p-1}u = 0 & \text{in } (0, 1), \\ u(0) = u(1) = 0 \end{cases}$$

and it has the following properties:

- (i) $J(0) = 0$, $J(u)$ is even in u , bounded from below and coercive;
- (ii) For any $k \in \mathbf{N}$, there exists a compact subset $A \subset E \setminus \{0\}$, which is symmetric with respect to 0, such that

$$\gamma(A) \geq k, \quad \max_{u \in A} J(u) < 0.$$

Actually, for any k -dimensional subspace $H \subset E$,

$$A = \{u \in H; \|u\|_E = \rho\}$$

with small $\rho > 0$ gives the desired compact set.

- (iii) $J(u)$ satisfies $(PS)_c$ for all $c \in \mathbf{R}$.

We define $I(u) : E \rightarrow \mathbf{R}$ by

$$I(u) = \begin{cases} 1 - \cos(2\pi \|u\|_E^2) & \text{for } \|u\|_E \leq 1, \\ J((\|u\|_E^2 - 1)u) & \text{for } \|u\|_E > 1, \end{cases}$$

Then we have

Proposition 3.4. *$I(u) : E \rightarrow \mathbf{R}$ satisfies the assumptions of Theorem 1.1. However 0 is an isolated critical point and the conclusion of Theorem 1.4 does not hold.*

Proof. Clearly $I(u)$ is even, bounded from below and coercive. Moreover $I(u)$ also satisfies $(PS)_c$ for all $c < 0$. In fact, if $(u_j)_{j=1}^\infty$ satisfies $I(u_j) \rightarrow c < 0$ and $\|I'(u_j)\| \rightarrow 0$, then we can easily see that $(u_j)_{j=1}^\infty$ is bounded as $j \rightarrow \infty$ and, after taking a subsequence, we may assume that $\|u_j\|_E \rightarrow d$ for some $d > 1$. Using this fact, we can see that $(u_j)_{j=1}^\infty$ has a strongly convergent subsequence. (Since all points on the unit sphere $S = \{x \in E; \|x\|_E = 1\}$ are critical points of $I(u)$ with critical value 0 and S is not compact, we note that $(PS)_0$ fails.) Thus $I(u)$ satisfies (A1) and (A2). We can see that (A3) holds easily. In fact, for any k -dimensional subspace $H \subset E$, choosing $\rho > 0$ small,

$$A = \{u \in E; \|u\|_E = 1 + \rho\}$$

satisfies $\gamma(A) \geq k$ and $\sup_{u \in A} I(u) < 0$. Thus $I(u)$ satisfies the assumptions of Theorem 1.1.

Clearly 0 is an isolated critical point of $I(u)$ and $I(u)$ does not have a sequence of critical points with (1.2)–(1.4). ■

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